Electromagnetic effects of neutrinos in an electron gas

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We study the electromagnetic properties of a system that consists of an electron background and a neutrino gas that may be moving or at rest, as a whole, relative to the background. The photon self-energy for this system is characterized by the usual transverse and longitudinal polarization functions, and two additional ones which are the focus of our calculations, that give rise to birefringence and anisotropic effects in the photon dispersion relations. Expressions for them are obtained, which depend on the neutrino number densities and involve momentum integrals over the electron distribution functions, and are valid for any value of the photon momentum and general conditions of the electron gas. Those expressions are evaluated explicitly for several special cases and approximations which are generally useful in astrophysical and cosmological settings. Besides studying the photon dispersion relations, we consider the macroscopic electrodynamic equations for this system, which involve the standard dielectric and permeability constants plus two additional ones related to the photon self-energy functions. As an illustration, the equations are used to discuss the evolution of a magnetic field perturbation in such a medium. This particular phenomena has also been considered in a recent work by Semikoz and Sokoloff as a mechanism for the generation of large-scale magnetic fields in the Early Universe as a consequence of the neutrino-plasma interactions, and allows us to establish contact with a specific application in a well defined context, with a broader scope and from a very different point of view.

I. INTRODUCTION AND SUMMARY

This work is concerned with the electromagnetic properties of a medium that consists of a matter background, such as an electron plasma, and a neutrino gas that moves, as a whole, relative to the matter background. Technically, the quantity of interest to us is the photon self-energy, from which the dispersion relations of the photon modes that propagate in the medium can be obtained, and from which other macroscopic quantities of physical interest can be determined.

Some aspects of this composite system were studied in Ref. [1] using the methods of real-time finite temperature field theory (FTFT)[28] which, from a modern point of view, provides a natural setting for studying the problems related to the propagation of photons in a medium. Largely stimulated by the work of Weldon[2, 3, 4], a convenient technique employed in FTFT is to carry out the calculations in a manifestly covariant form. As is by now familiar, this is implemented by introducing the velocity four-vector u^μ of the medium, in terms of which the thermal propagators are written in a covariant form. In this way, covariance is maintained, but quantities such as the photon self-energy depend on the vector u^μ in addition to the kinematic momentum variables of the problem. Generally, for practical purposes the vector u^μ is set to $(1, \vec{0})$ in the end, which is equivalent to having carried out the calculation in the rest frame of the medium from the beginning, and this is usually the relevant physical situation.

However, as noted in Ref. [1], the system we are considering provides an example for a novel application of FTFT. A distinctive feature of this system is that the matter background on one hand, and the neutrino gas on the other, each is characterized by its own velocity four-vector. Thus, if we denote by u^μ the velocity four-vector of the matter background, and by v^μ the corresponding one for the neutrino gas, then we can take the matter background to be at rest, so that

$$u^\mu = (1, \vec{0}) \,, \tag{1.1}$$

but we must keep

$$v^\mu = (v^0, \vec{V}) \,. \tag{1.2}$$

Therefore, the physical quantities such as the photon self-energy depend on the momentum variables and u^μ as usual, and in addition on the vector v^μ. This additional dependence can have physical effects that cannot be produced by the stationary background alone.
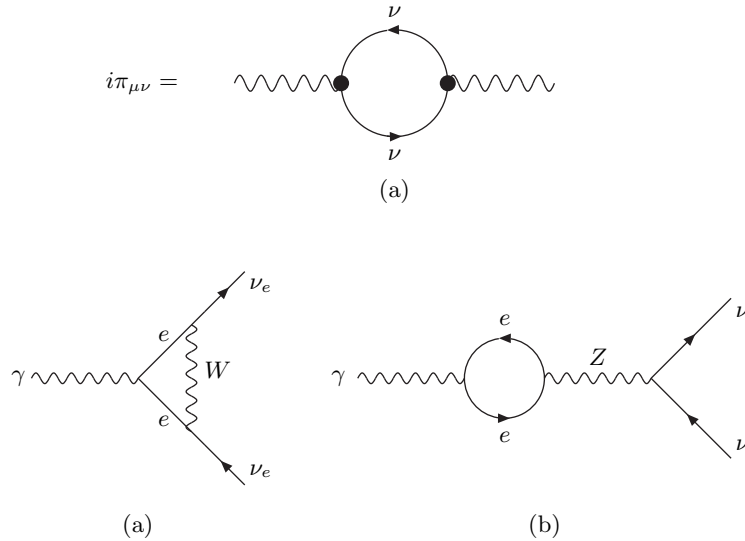
(a)

(a)   (b)

FIG. 1: A class of diagrams that contribute to the photon self-energy in the presence of a neutrino background. The filled circle in diagram (a) represents the neutrino electromagnetic vertex that is induced by the interactions with the electron background[8, 9, 10, 11, 12, 13], which can be determined by calculating the off-shell amplitude corresponding to diagrams (b) and (c)[14].
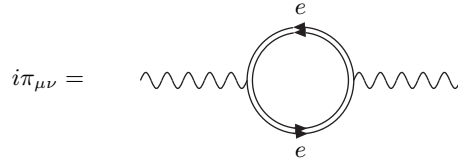

FIG. 2: A class of diagrams that contribute to the photon self-energy in the presence of a neutrino background. The double electron line represents the thermal electron propagator, with the effects of the neutrino gas taken into account.

The focus of attention in Ref. [1] was the effect that the collective neutrino-plasma interactions could have on the stability of such systems[5, 6], in analogy with the *stream instabilities* that are familiar in plasma physics research, examples of which are discussed in many textbooks[7]. The calculation was focused on the contribution to the photon self-energy from the set of diagrams represented in Fig. 1.

On the other hand, it was recognized some time ago[15, 16, 17, 18] that the presence of neutrinos induces *birefringence* effects in the medium, similar to those that exist in materials that exhibit *natural optical activity*[19]. These effects are a consequence of a new term, denoted by π_P in Ref. [15], that shows up in the photon self-energy, which is proportional to the neutrino-antineutrino asymmetry and is odd under various discrete space-time symmetries. However, the previous calculations of these effects have taken into account only the neutrino background. In particular, for the purpose of determining π_P in Ref. [17], the background was taken to consist only of neutrinos and antineutrinos. Therefore, for all the other particles the vacuum propagator was used in the calculation.

Here we note that, in the presence of neutrinos, there is an important contribution to the photon self-energy which is proportional to both the neutrino asymmetry and the electron number density. It is best specified by referring to the diagram depicted in Fig. 2. As indicated by the double line in this diagram, the electron propagator to be used in the calculation is the *dressed* propagator with the correct electron self-energy in the medium. The electron self-energy includes the effect of the neutrino gas, as represented by the diagrams shown in Fig. 3. As we show, this
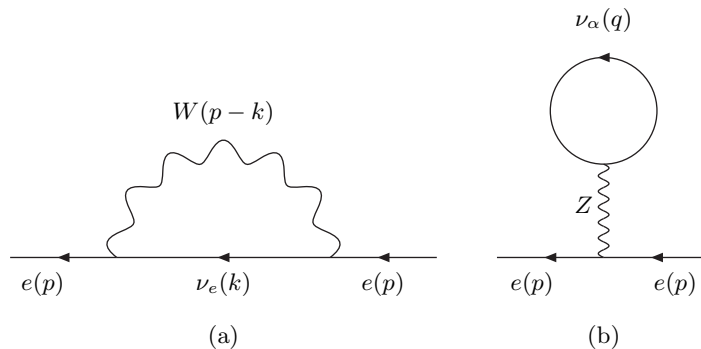
FIG. 3: One-loop diagrams for the electron self-energy in a neutrino gas.

contribution to the photon self-energy is of no consequence for the instability issues, but it yields a novel contribution to the optical activity of the system, that can lead to interesting and significant physical effects. In particular, we note that the contribution to π_P calculated in Ref. [17] is proportional to the photon momentum squared q^2. In contrast, the contribution that is determined here has no such term, and it is the only one that survives in the $q \to 0$ limit. This feature leads to effects that can manifest themselves at a macroscopic level, in the static and long wavelength electromagnetic regime.

In this work we compute the contribution to the photon self-energy that is proportional to both the neutrino-antineutrino asymmetry and the electron density, and consider the consequences for the propagation of photons and for the electromagnetic properties of the medium. The calculation is based on the application of FTFT to calculate the photon self-energy diagram shown in Fig. 2, using the electron propagator that includes the effect of the neutrino-electron interactions. The implicit assumption is that, in its own rest frame, the neutrino gas has a momentum distribution function that is parametrized in the usual way. The presence of the neutrinos gives rise to two additional polarization functions besides the ordinary longitudinal and transverse polarization functions the photon self-energy, that we denote by π_P and π'_P. A non-zero value of π_P by itself leads to optical activity effects, while π'_p induces anisotropic effects. Aside from the implications for the photon dispersion relations, the results for the photon self-energy can be interpreted in terms of the macroscopic electromagnetic properties of the system. In particular, the effects due to π_P and π'_P remain finite in the limit $q \to 0$, and therefore they can manifest themselves in macroscopic effects in the long wavelength and static regimes. As a specific application, we consider the evolution of a magnetic field perturbation in the system, and we arrive at an equation for the dynamics of the magnetic field that has been suggested by Semikoz and Sokoloff[20], in the context of a mechanism for the generation of large-scale magnetic fields in the Early Universe as a consequence of the neutrino-plasma interactions. Thus, besides extending the earlier calculations already mentioned, the present work makes contact and complements this recent work, which is based on a treatment using the kinetic equations of the neutrino-plasma system. The results and formulas we present are applicable in a variety of situations where the neutrino interactions with the other background particles are important, and the method we employ could also be useful in the study of similar problems that may arise in other contexts.

A word about the strategy of our calculation is in order. In principle, the diagrams involved are numerous, and many of them are shown in Ref. [17]. However our calculation is simplified for the following reason. Since we are interested in the terms that contain both the neutrino and the electron distribution functions, the momentum integrations are effectively cutoff. To order $1/M_W^2$, we can then replace the W, Z boson propagators by their local limit, so that the neutrino-electron interactions can be approximated by the local Fermi interactions. In that limit, the set of diagrams that are relevant to extract the contribution that we are seeking, to order $1/M_W^2$, collapses to the class of diagram represented in Fig. 2, which we are considering in the manner we have indicated.

We begin, in Section II by writing down the expression for the electron propagator that will be used in the calculation of the photon self-energy, which includes the thermal effects as well as the effects of the neutrino-electron interactions. In Section III the self-energy tensor is calculated and in terms of the usual transverse and longitudinal components and the two additional components that we have mentioned, and the integral formulas for the latter are obtained and evaluated for some cases in Section IV. The photon dispersion relations for this system are considered in Section V, the application to the study of the macroscopic electromagnetic properties of this system is in Section VI, and Section VII contains our conclusions.

II.   ELECTRON PROPAGATOR

The electron self-energy function $\Sigma_e(p)$ is determined by computing the diagrams depicted in Fig. 3. A straight-forward calculation, using the thermal propagator for the internal neutrino lines, yields to order $1/M_W^2$

$$\Sigma_e = \not{p}(\lambda_V + \lambda_A \gamma_5)\,, \tag{2.1}$$

where

$$
\begin{aligned}
\lambda_V &= \sqrt{2}G_F\left[a_e^{(Z)}\sum_{\alpha=e,\mu,\tau}(n_{\nu_\alpha}-n_{\bar{\nu}_\alpha})+n_{\nu_e}-n_{\bar{\nu}_e}\right]\,, \\
\lambda_A &= \sqrt{2}G_F\left[b_e^{(Z)}\sum_{\alpha=e,\mu,\tau}(n_{\nu_\alpha}-n_{\bar{\nu}_\alpha})-(n_{\nu_e}-n_{\bar{\nu}_e})\right]\,.
\end{aligned} \tag{2.2}
$$

Here the coefficients $a_e^{(Z)}, b_e^{(Z)}$ are the vector and axial vector neutral-current couplings of the electron,

$$
\begin{aligned}
a_e^{(Z)} &= -\frac{1}{2}+2\sin^2\theta_W \\
b_e^{(Z)} &= \frac{1}{2}\,,
\end{aligned} \tag{2.3}
$$

and the $n_{\nu_\alpha}, n_{\bar{\nu}_\alpha}$ denote respectively the total number density of each neutrino or antineutrino specie in the medium.

The electron propagator to be used in our calculation of the photon self-energy is given by

$$S_e(p) = S_F(p) - \left[S_F(p) - \bar{S}_F(p)\right]\eta_e(p\cdot u)\,, \tag{2.4}$$

where $\bar{S}_F = \gamma^0 S_F^\dagger \gamma^0$, and

$$\eta_e(p) = \theta(p\cdot u)f_e(p\cdot u) + \theta(-p\cdot u)f_{\bar{e}}(-p\cdot u)\,, \tag{2.5}$$

with

$$
\begin{aligned}
f_e(x) &= \frac{1}{e^{\beta(x-\mu_e)}+1} \\
f_{\bar{e}}(x) &= \frac{1}{e^{\beta_e(x+\mu_e)}+1}\,.
\end{aligned} \tag{2.6}
$$

Here β_e is the inverse temperature and μ_e the chemical potential of the electron gas. In addition, S_F is the electron propagator in the presence of the neutrino gas, which is given by

$$S_F^{-1}(p) = S_0^{-1} - \Sigma_e(p)\,, \tag{2.7}$$

where S_0 is the free propagator in the vacuum

$$S_0 = \frac{\not{p}+m_e}{p^2-m_e^2+i\epsilon}\,, \tag{2.8}$$

and Σ_e has been given in Eq. (2.1). Therefore, to leading oder in $1/M_W^2$,

$$S_F(p) = S_0(p) + S_0(p)\Sigma_e(p)S_0(p)\,. \tag{2.9}$$

Substituting this in Eq. (2.4), we obtain the electron thermal propagator to be used in our computation

$$S_e = S_0 + S_T + S' + S_T'\,, \tag{2.10}$$

where S_0 is given in Eq. (2.8), S_T is the usual thermal part of the electron propagator

$$S_T(p) = 2\pi i\delta(p^2-m_e^2)\eta_e(p\cdot u)(\not{p}+m_e)\,, \tag{2.11}$$

while

$$S'(p) = \frac{(\not{p}+m_e)\Sigma_e(p)(\not{p}+m_e)}{(p^2-m_e^2+i\epsilon)^2}\,, \tag{2.12}$$

and

$$S'_T(p) = -2\pi i \delta'(p^2 - m_e^2)\eta_e(p \cdot u)(\not{p} + m_e)\Sigma_e(p)(\not{p} + m_e) \,. \tag{2.13}$$

In Eq. (2.13), δ' denotes the derivative of the delta function with respect to its argument.

In some respects, the present calculation resembles the calculation of the photon self-energy in an electron background in the presence of a magnetic field[21, 22]. In that case, the calculation involves the use of the thermal generalization of the Schwinger propagator, which takes into account the B field and the thermal effects of the electron background. In the present case, the neutrino current acts as the external field, playing the role of the B field in the former case.

III. PHOTON SELF-ENERGY

We decompose the photon self-energy into various parts according to whether or not they depend on the neutrino densities. Therefore, discarding the term that is independent of the particle densities, we write

$$\pi_{\mu\nu} = \pi_{\mu\nu}^{(e)} + \pi_{\mu\nu}^{(\nu)} + \pi_{\mu\nu}^{(e\nu)} \,, \tag{3.1}$$

where $\pi_{\mu\nu}^{(e)}$ is the purely electronic contribution, while $\pi_{\mu\nu}^{(\nu)}$, which depends on the neutrino distributions but not on the electron distribution, corresponds to the contribution to the photon self-energy that was computed in Ref. [17]. Although the results for $\pi_{\mu\nu}^{(\nu)}$, and of course $\pi_{\mu\nu}^{(e)}$ are known, we state them below in the form that will be useful for later reference. On the other hand, $\pi_{\mu\nu}^{(e\nu)}$ contains the terms that depend on both the neutrino and electron distributions, and is the focus of the present paper. To determine it, we start from the expression corresponding to the diagram of Fig. 2,

$$i\pi'_{\mu\nu} \equiv -(-ie)^2 \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left[\gamma_\mu i S_e(p+q)\gamma_\nu i S_e(p) \right] \,, \tag{3.2}$$

where $S_e(p)$ is given by Eq. (2.4), with S_F determined from Eq. (2.7). Substituting Eq. (2.10) into Eq. (3.2) and singling out the terms that contain the electron distributions as well as the neutrino densities we obtain,

$$\begin{aligned} i\pi_{\mu\nu}^{(e\nu)} &= e^2 \int \frac{d^4p}{(2\pi)^4} \{ \text{Tr}\,\gamma_\mu i S_0(p')\gamma_\nu i S'_T(p) + \text{Tr}\,\gamma_\mu i S'_T(p')\gamma_\nu i S_0(p) \\ &\quad + \text{Tr}\,\gamma_\mu i S'(p')\gamma_\nu i S_T(p) + \text{Tr}\,\gamma_\mu i S_T(p')\gamma_\nu i S'(p) \} \,, \end{aligned} \tag{3.3}$$

where

$$p' = p + q \,. \tag{3.4}$$

A. $\pi_{\mu\nu}^{(e)}$

The pure electron term is given by

$$i\pi_{\mu\nu}^{(e)} = e^2 \int \frac{d^4p}{(2\pi)^4} \{ \text{Tr}\,\gamma_\mu i S_0(p')\gamma_\nu i S_T(p) + \text{Tr}\,\gamma_\mu i S_T(p')\gamma_\nu i S_0(p) \} \,. \tag{3.5}$$

Computing the traces and carrying out the p^0 integration, $\pi_{\mu\nu}^{(e)}$ can be written in the form

$$\pi_{\mu\nu}^{(e)} = -4e^2 \int \frac{d^3p}{(2\pi)^3 2E}(f_e + f_{\bar{e}}) \left\{ \frac{L_{\mu\nu}}{q^2 + 2p \cdot q} + (q \to -q) \right\} \,, \tag{3.6}$$

where

$$L_{\mu\nu} = 2p_\mu p_\nu + p_\mu q_\nu + q_\mu p_\nu - p \cdot q g_{\mu\nu} \,. \tag{3.7}$$

The fact that $\pi_{\mu\nu}^{(e)}$ is symmetric and satisfies $q^\mu \pi_{\mu\nu}^{(e)} = q^\nu \pi_{\mu\nu}^{(e)} = 0$ implies that it is of the form

$$\pi_{\mu\nu}^{(e)} = \pi_T^{(e)} R_{\mu\nu} + \pi_L^{(e)} Q_{\mu\nu} \,, \tag{3.8}$$

where

$$R_{\mu\nu} = \tilde{g}_{\mu\nu} - Q_{\mu\nu},$$
$$Q_{\mu\nu} = \frac{\tilde{u}_\mu \tilde{u}_\nu}{\tilde{u}^2}, \tag{3.9}$$

with

$$\tilde{u}_\mu \equiv \tilde{g}_{\mu\nu} u^\nu, \tag{3.10}$$

and

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}. \tag{3.11}$$

In general $\pi_{T,L}^{(e)}$ are functions of the scalar variables

$$\omega = q \cdot u$$
$$Q = \sqrt{\omega^2 - q^2}, \tag{3.12}$$

which have the interpretation of being the photon energy and momentum, in the rest frame of the electron gas. The functions $\pi_{T,L}^{(e)}$ are determined by projecting Eq. (3.6) with the tensors $R_{\mu\nu}$ and $Q_{\mu\nu}$. This procedure then leads to

$$\pi_T^{(e)} = -2e^2 \left(A_e + \frac{q^2}{Q^2} B_e \right),$$
$$\pi_L^{(e)} = 4e^2 \frac{q^2}{Q^2} B_e \tag{3.13}$$

where

$$A_e = \int \frac{d^3p}{(2\pi)^3 2E} (f_e + f_{\bar{e}}) \left[\frac{2m_e^2 - 2p \cdot q}{q^2 + 2p \cdot q} + (q \to -q) \right],$$
$$B_e = \int \frac{d^3p}{(2\pi)^3 2E} (f_e + f_{\bar{e}}) \left[\frac{2(p \cdot u)^2 + 2(p \cdot u)(q \cdot u) - p \cdot q}{q^2 + 2p \cdot q} + (q \to -q) \right]. \tag{3.14}$$

While these formulas can be used to evaluate the electronic contribution to the photon self-energy in various situations, we need not proceed any further in that direction since, as we have already mentioned, the results are well known and we can simply quote the relevant ones when we need them.

B. $\pi_{\mu\nu}^{(\nu)}$

In addition to the tensors $R_{\mu\nu}$ and $Q_{\mu\nu}$ that are defined in Eq. (3.9), we introduce

$$P_{\mu\nu} = \frac{i}{Q} \epsilon_{\mu\nu\alpha\beta} q^\alpha u^\beta,$$
$$P'_{\mu\nu} = \frac{i}{Q} \epsilon_{\mu\nu\alpha\beta} q^\alpha v'^\beta, \tag{3.15}$$

with

$$v'^\mu = v^\mu - u^\mu (u \cdot v). \tag{3.16}$$

Then the results of Ref. [17] can be expressed in the form

$$\pi_{\mu\nu}^{(\nu)} = \pi_P^{(\nu)} (P'_{\mu\nu} + u \cdot v P_{\mu\nu}), \tag{3.17}$$

where

$$\pi_P^{(\nu)} = \frac{e^2 G_F}{\sqrt{2}\pi^2} \left(\frac{q^2}{m_e^2} \right) Q(n_{\nu_e} - n_{\bar{\nu}_e}) J(q^2). \tag{3.18}$$

$J(q^2)$ is given explicitly in that reference, but its precise value will not be relevant here. We only wish to note the presence of the kinematic factor of q^2 in Eq. (3.18). For the purpose of determining the dispersion relations, this factor can be set equal to the plasma frequency squared ω_p^2, since in the lowest order $q^2 \sim \omega_p^2$. However, in other applications, such the one as we will consider in Section VI B, the appropriate kinematic regime corresponds to taking $q \to 0$, and in those cases $\pi_P^{(\nu)}$ is not relevant.

C. Evaluation of $\pi_{\mu\nu}^{(e\nu)}$

Computing the traces and carrying out the p^0 integration in Eq. (3.3), $\pi_{\mu\nu}^{(e\nu)}$ can be expressed in the form

$$\pi_{\mu\nu}^{(e\nu)} = (-4e^2)(\lambda_V T_{\mu\nu}^{(V)} + \lambda_A T_{\mu\nu}^{(A)}),\tag{3.19}$$

where

$$
\begin{aligned}
T_{\mu\nu}^{(V)} &= \int \frac{d^4p}{(2\pi)^3}\eta_e(p\cdot u)\left\{\frac{-L_{\mu\nu}^{(1)}\delta'(p^2-m_e^2)}{d} + \frac{L_{\mu\nu}^{(2)}\delta(p^2-m_e^2)}{d^2} + (q\to -q)\right\} \\
T_{\mu\nu}^{(A)} &= i\epsilon_{\mu\nu\alpha\beta}\int \frac{d^4p}{(2\pi)^3}\eta_e(p\cdot u)\left\{\frac{-K^{(1)\alpha\beta}\delta'(p^2-m_e^2)}{d} + \frac{K^{(2)\alpha\beta}\delta(p^2-m_e^2)}{d^2} - (q\to -q)\right\},
\end{aligned}\tag{3.20}
$$

with

$$
\begin{aligned}
L_{\mu\nu}^{(1)} &= (m_e^2 - p^2)\left[p'_\mu v_\nu - g_{\mu\nu}p'\cdot v + v_\mu p'_\nu\right] + (2p\cdot v)m_e^2 g_{\mu\nu} + 2p\cdot v\left[p'_\mu p_\nu + p_\mu p'_\nu - p\cdot p' g_{\mu\nu}\right] \\
L_{\mu\nu}^{(2)} &= (m_e^2 - p'^2)\left[p_\mu v_\nu - g_{\mu\nu}p\cdot v + v_\mu p_\nu\right] + (2p'\cdot v)m_e^2 g_{\mu\nu} + 2p'\cdot v\left[p'_\mu p_\nu + p_\mu p'_\nu - p\cdot p' g_{\mu\nu}\right] \\
K_{\alpha\beta}^{(1)} &= (p^2 - m_e^2)p_\alpha v_\beta + (p^2 + m_e^2)q_\alpha v_\beta - 2p\cdot v\, q_\alpha p_\beta, \\
K_{\alpha\beta}^{(2)} &= (m_e^2 - p'^2)p_\alpha v_\beta + 2m_e^2\, q_\alpha v_\beta - 2p'\cdot v\, q_\alpha p_\beta,
\end{aligned}\tag{3.21}
$$

and

$$d = (p+q)^2 - m_e^2.\tag{3.22}$$

1. Evaluation of $T_{\mu\nu}^{(V)}$

The integral expression for $T_{\mu\nu}^{(V)}$ can be simplified as follows. Defining

$$
\begin{aligned}
C_{\mu\nu} &= p_\mu v_\nu + v_\mu p_\nu - p\cdot v g_{\mu\nu}, \\
D_{\mu\nu} &= (m_e^2 - p\cdot p')g_{\mu\nu} + p_\mu p'_\nu + p'_\mu p_\nu,
\end{aligned}\tag{3.23}
$$

$L_{\mu\nu}^{(1,2)}$ can be expressed in the form

$$
\begin{aligned}
L_{\mu\nu}^{(1)} &= (m_e^2 - p^2)\left[v^\lambda\partial_\lambda D_{\mu\nu} - C_{\mu\nu}\right] + 2p\cdot v D_{\mu\nu}, \\
L_{\mu\nu}^{(2)} &= -dC_{\mu\nu} + 2p'\cdot v D_{\mu\nu},
\end{aligned}\tag{3.24}
$$

where d is given in Eq. (3.22), $\partial_\mu \equiv \partial/\partial p^\mu$, and we have used the following relation,

$$p'_\mu v_\nu + v_\mu p'_\nu - p'\cdot v g_{\mu\nu} = v^\lambda\partial_\lambda D_{\mu\nu} - C_{\mu\nu},\tag{3.25}$$

which can be verified by explicit computation. Using the relations

$$
\begin{aligned}
(p^2 - m_e^2)\delta'(p^2-m_e^2) &= -\delta(p^2-m_e^2), \\
v^\mu\partial_\mu\delta(p^2-m_e^2) &= 2p\cdot v\delta'(p^2-m_e^2),
\end{aligned}\tag{3.26}
$$

it then follows that

$$L_{\mu\nu}^{(1)}\delta'(p^2-m_e^2) = v^\lambda\partial_\lambda\left[D_{\mu\nu}\delta(p^2-m_e^2)\right] - C_{\mu\nu}\delta(p^2-m_e^2),\tag{3.27}$$

and similarly

$$\frac{L_{\mu\nu}^{(2)}}{d^2} = -D_{\mu\nu}v^\lambda\partial_\lambda\left(\frac{1}{d}\right) - \frac{C_{\mu\nu}}{d},\tag{3.28}$$

where we have used

$$v^\lambda \partial_\lambda \left(\frac{1}{d} \right) = -2p' \cdot v \frac{1}{d^2} \, . \tag{3.29}$$

Therefore, from Eq. (3.20), with the help of Eqs. (3.27) and (3.28),

$$T_{\mu\nu}^{(V)} = \int \frac{d^4p}{(2\pi)^3} \eta_e(p \cdot u) \left\{ -v^\lambda \partial_\lambda \left[\frac{D_{\mu\nu}\delta(p^2 - m_e^2)}{d} \right] + (q \to -q) \right\} \, , \tag{3.30}$$

which, after a partial integration and then carrying out the integration over p^0 using the delta function, yields

$$T_{\mu\nu}^{(V)} = v \cdot u \int \frac{d^3p}{(2\pi)^3 2E} \frac{\partial(f_e - f_{\bar{e}})}{\partial E} \left\{ \frac{L_{\mu\nu}}{q^2 + 2p \cdot q} + (q \to -q) \right\} \, , \tag{3.31}$$

with $L_{\mu\nu}$ given in Eq. (3.7). This has the same structure as the normal electron background contribution, given in Eq. (3.6). Therefore, the same arguments used in Section III A can be applied here to conclude that $T_{\mu\nu}^{(V)}$ can be expressed in the form

$$T_{\mu\nu}^{(V)} = \frac{1}{2} \left(A'_e + \frac{q^2}{Q^2} B'_e \right) R_{\mu\nu} + \left(\frac{-q^2}{Q^2} \right) B'_e Q_{\mu\nu} \, , \tag{3.32}$$

where

$$
\begin{aligned}
A'_e &= u \cdot v \int \frac{d^3p}{(2\pi)^3 2E} \frac{\partial(f_e + f_{\bar{e}})}{\partial E} \left[\frac{2m_e^2 - 2p \cdot q}{q^2 + 2p \cdot q} + (q \to -q) \right] \, , \\
B'_e &= u \cdot v \int \frac{d^3p}{(2\pi)^3 2E} \frac{\partial(f_e + f_{\bar{e}})}{\partial E} \left[\frac{2(p \cdot u)^2 + 2(p \cdot u)(q \cdot u) - p \cdot q}{q^2 + 2p \cdot q} + (q \to -q) \right] \, .
\end{aligned}
\tag{3.33}
$$

2. Evaluation of $T_{\mu\nu}^{(A)}$

From Eq. (3.21), using Eq. (3.26) we can write

$$\frac{-K_{\alpha\beta}^{(1)}\delta'(p^2 - m_e^2)}{d} + \frac{K_{\alpha\beta}^{(2)}\delta(p^2 - m_e^2)}{d^2} = 2m_e^2 q_\alpha v_\beta \left[\frac{\delta(p^2 - m_e^2)}{d^2} - \frac{\delta'(p^2 - m_e^2)}{d} \right] + v^\lambda \partial_\lambda \left(\frac{q_\alpha p_\beta \delta(p^2 - m_e^2)}{d} \right) \, . \tag{3.34}$$

Therefore,

$$T_{\mu\nu}^{(A)} = i\epsilon_{\mu\nu\alpha\beta} q^\alpha \left[I_1^\beta + I_2 v^\beta \right] \, , \tag{3.35}$$

where

$$
\begin{aligned}
I_1^\beta &= \int \frac{d^4p}{(2\pi)^3} \eta_e(p \cdot u) \left\{ v^\lambda \partial_\lambda \left[\frac{p^\beta \delta(p^2 - m_e^2)}{d} \right] \right\} + (q \to -q) \, , \\
I_2 &= 2m_e^2 \int \frac{d^4p}{(2\pi)^3} \eta_e(p \cdot u) \left\{ \frac{\delta(p^2 - m_e^2)}{d^2} - \frac{\delta'(p^2 - m_e^2)}{d} \right\} + (q \to -q) \, .
\end{aligned}
\tag{3.36}
$$

For I_1^β, by partial integration we obtain

$$I_1^\beta = -v \cdot u \int \frac{d^3p}{(2\pi)^3 2E} \frac{\partial(f_e + f_{\bar{e}})}{\partial E} p^\beta \left[\frac{1}{q^2 + 2p \cdot q} + (q \to -q) \right] \, . \tag{3.37}$$

Since the integral is a function only of the vectors q^μ and u^μ, it must be of the form

$$I_1^\beta = I_1 u^\beta + I'_1 q^\beta \, . \tag{3.38}$$

Substituting it in Eq. (3.35),

$$T_{\mu\nu}^{(A)} = i\epsilon_{\mu\nu\alpha\beta} q^\alpha \left[I_1 u^\beta + I_2 v^\beta \right] \, , \tag{3.39}$$

and therefore I_1' is not relevant. On the other hand, I_1 can be determined by using the projection relation

$$\tilde{u}_\beta I_1^\beta = \tilde{u}^2 I_1 \,, \tag{3.40}$$

which, from Eq. (3.37), yields the formula

$$I_1 = \frac{v \cdot u}{Q^2} \int \frac{d^3 p}{(2\pi)^3 2E} \frac{\partial(f_e + f_{\bar{e}})}{\partial E} \left[\frac{q^2 (p \cdot u) - (q \cdot u)(p \cdot q)}{q^2 + 2p \cdot q} \right] + (q \to -q) \,. \tag{3.41}$$

To carry out the integration over p^0 in Eq. (3.36) we write

$$\delta'(p^2 - m_e^2) = \frac{1}{2p^0} \frac{\partial}{\partial p^0} \delta(p^2 - m_e^2) \,, \tag{3.42}$$

and using the usual rule for the integration over the derivative of the delta function after some algebra we obtain

$$I_2 = 2m_e^2 \int \frac{d^3 p}{(2\pi)^3 2E} \frac{\partial}{\partial E} \left[\frac{1}{2E} \left(\frac{f_e + f_{\bar{e}}}{q^2 + 2p \cdot q} \right) \right] + (q \to -q) \,. \tag{3.43}$$

D. Summary

From the results we have obtained, it follows that the photon self-energy can be expressed in the form

$$\pi_{\mu\nu} = \pi_T R_{\mu\nu} + \pi_L Q_{\mu\nu} + \pi_P P_{\mu\nu} + \pi_P' P_{\mu\nu}' \,, \tag{3.44}$$

where $P_{\mu\nu}$ and $P_{\mu\nu}'$ have been defined in Eq. (3.15). Using Eqs. (3.8) and (3.19), together with the results for $T_{\mu\nu}^{(V,A)}$ given in Eqs. (3.32) and (3.39), the coefficients in Eq. (3.44) are then given by

$$\pi_T = \pi_T^{(e)} - 2e^2 \lambda_V \left(A_e' + \frac{q^2}{Q^2} B_e' \right) \,,$$

$$\pi_L = \pi_L^{(e)} + 4e^2 \lambda_V \left(\frac{q^2}{Q^2} \right) B_e' \,,$$

$$\pi_P = (u \cdot v) \pi_P^{(\nu)} + \pi_P^{(e\nu)}$$

$$\pi_P' = \pi_P^{(\nu)} + \pi_P'^{(e\nu)} \tag{3.45}$$

where

$$\pi_P^{(e\nu)} = -4e^2 \lambda_A Q[I_1 + (u \cdot v)I_2] \,,$$

$$\pi_P'^{(e\nu)} = -4e^2 \lambda_A Q I_2 \,, \tag{3.46}$$

with $I_{1,2}$ given by Eqs. (3.41) and (3.43), respectively.

The terms proportional to λ_V give only small correction to the pure electronic terms. For example, consider the case of a classical electron distribution, for which we can use

$$\frac{\partial f_{e,\bar{e}}}{\partial E} \approx -\beta f_{e,\bar{e}} \,. \tag{3.47}$$

Then from Eq. (3.33),

$$A_e' = -\beta A_e \,,$$

$$B_e' = -\beta B_e \,, \tag{3.48}$$

and remembering Eq. (3.13), it follows that the neutrino-dependent contribution to $\pi_{T,L}$ is smaller than the electron term by a factor of $\beta \lambda_V$. Assuming, for illustrative purposes, that the neutrino gas can also be treated classically, so that $n_\nu \sim T^3$, then $\beta \lambda_V \sim G_F T^2$, which is negligible in most situations of interest. Therefore, for all practical purposes of interest to us, we will neglect that contribution in Eq. (3.45) and use

$$\pi_T = \pi_T^{(e)} \,,$$

$$\pi_L = \pi_L^{(e)} \tag{3.49}$$

in what follows.

IV. EVALUATION OF $\pi_P^{(e\nu)}$ AND $\pi_P'^{(e\nu)}$

Useful formulas for $\pi_P^{(e\nu)}$ and $\pi_P'^{(e\nu)}$ can be obtained by evaluating the integrals $I_{1,2}$ defined in Eqs. (3.41) and (3.43) in the long wavelength limit. The expressions for $I_{1,2}$ in that limit can be obtained by applying the auxiliary formula[23, 24]

$$\int \frac{d^3p}{(2\pi)^3} \frac{\mathcal{F}(p,q)}{[q^2 + 2p\cdot q]^n} = \int \frac{d^3p}{(2\pi)^3} \frac{\left(\mathcal{F} - \frac{\vec{Q}}{2}\cdot\frac{d\mathcal{F}}{d\vec{p}} - \frac{n\omega}{2E}\mathcal{F}\right)}{[2E\omega - 2\vec{p}\cdot\vec{Q}]^n}, \tag{4.1}$$

and they are valid for

$$\omega, Q \ll \langle \mathcal{E} \rangle, \tag{4.2}$$

where $\langle \mathcal{E} \rangle$ denotes a typical average energy of the electrons in the background. In Eq. (4.1) the symbol $\frac{d}{d\vec{p}}$ stands for the total momentum derivative,

$$\frac{d}{d\vec{p}} = \frac{\partial}{\partial\vec{p}} + \frac{\vec{p}}{E}\frac{\partial}{\partial E}. \tag{4.3}$$

Let us consider I_2 first. We rewrite Eq. (3.43) in the form

$$I_2 = \frac{m_e^2}{2} \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{f'}{q^2 + 2q\cdot p} - \frac{2\omega f}{[q^2 + 2q\cdot p]^2} \right\} + (q \to -q), \tag{4.4}$$

where, to simplify the notation, we have defined

$$\begin{aligned} f &= \frac{f_e + f_{\bar{e}}}{E^2}, \\ f' &= \frac{1}{E}\frac{\partial}{\partial E}\left(\frac{f_e + f_{\bar{e}}}{E}\right). \end{aligned} \tag{4.5}$$

Then by direct application of Eq. (4.1) we obtain

$$\begin{aligned} I_2 &= \frac{m_e^2}{4} \int \frac{d^3p}{(2\pi)^3} \frac{\omega}{E[\omega E - \vec{p}\cdot\vec{Q}]^2}\left[\vec{p}\cdot\vec{Q}\frac{\partial f}{\partial E} + 2\omega f\right] \\ &\quad - \frac{m_e^2}{4} \int \frac{d^3p}{(2\pi)^3} \frac{1}{E[\omega E - \vec{p}\cdot\vec{Q}]}\left[\vec{p}\cdot\vec{Q}\frac{\partial f'}{\partial E} + \omega f'\right]. \end{aligned} \tag{4.6}$$

Furthermore, denoting by $\bar{v}$ the average velocity of the particles in the background, the following approximate formulas are useful for practical applications,

$$I_2 = \begin{cases} \frac{m_e^2}{4} \int \frac{d^3p}{(2\pi)^3}\frac{1}{E}\frac{\partial}{\partial E}\left[\frac{1}{E}\frac{\partial}{\partial E}\left(\frac{f_e + f_{\bar{e}}}{E}\right)\right] & (\omega \ll \bar{v}Q) \\ -\frac{m_e^2}{4} \int \frac{d^3p}{(2\pi)^3}\frac{1}{E}\frac{\partial}{\partial E}\left(\frac{f_e + f_{\bar{e}}}{E^3}\right) & (\omega \gg \bar{v}Q), \end{cases} \tag{4.7}$$

which are obtained from Eq. (4.6) by taking the static limit $\omega = 0$, or the $\vec{Q} = 0$ limit, respectively. The remaining integration can be carried out simply for particular cases of the distribution functions, and we consider some of them below.

Regarding I_1, it can be written in the form

$$I_1 = \frac{u\cdot v}{2Q^2} \int \frac{d^3p}{(2\pi)^3} \frac{F_1}{q^2 + 2p\cdot q} + (q \to -q), \tag{4.8}$$

where

$$F_1 \equiv \frac{\partial(f_e + f_{\bar{e}})}{\partial E}\left[-Q^2 + \frac{\omega\vec{p}\cdot\vec{Q}}{E}\right], \tag{4.9}$$

and applying Eq. (4.1),

$$I_1 = -\frac{u \cdot v}{4Q^2} \int \frac{d^3p}{(2\pi)^3} \left(\vec{Q} \cdot \frac{dF_1}{d\vec{p}} + \frac{\omega}{E} F_1 \right) \frac{1}{E\omega - \vec{p} \cdot \vec{Q}} . \tag{4.10}$$

To compute the momentum derivative we use

$$\frac{\partial F_1}{\partial E} = -Q^2 \frac{\partial^2 (f_e + f_{\bar{e}})}{\partial E^2} + \omega \vec{p} \cdot \vec{Q} \frac{\partial}{\partial E} \left[\frac{1}{E} \frac{\partial (f_e + f_{\bar{e}})}{\partial E} \right] \tag{4.11}$$

which, using $\vec{p} \cdot \vec{Q} = E\omega - (E\omega - \vec{p} \cdot \vec{Q})$, we write as

$$\frac{\partial F_1}{\partial E} = -Q^2 \frac{\partial^2 (f_e + f_{\bar{e}})}{\partial E^2} + \omega^2 E \frac{\partial}{\partial E} \left[\frac{1}{E} \frac{\partial (f_e + f_{\bar{e}})}{\partial E} \right] - \omega (E\omega - \vec{p} \cdot \vec{Q}) \frac{\partial}{\partial E} \left[\frac{1}{E} \frac{\partial (f_e + f_{\bar{e}})}{\partial E} \right] . \tag{4.12}$$

In addition

$$\frac{\partial F_1}{\partial \vec{p}} = \frac{\omega}{E} \frac{\partial (f_e + f_{\bar{e}})}{\partial E} \vec{Q} , \tag{4.13}$$

and therefore, remembering Eq. (4.3),

$$\frac{\omega}{E} F_1 + \vec{Q} \cdot \frac{dF_1}{d\vec{p}} = q^2 \frac{\vec{p} \cdot \vec{Q}}{E} \frac{\partial^2 (f_e + f_{\bar{e}})}{\partial E^2} - \omega \frac{\vec{p} \cdot \vec{Q}}{E} (E\omega - \vec{p} \cdot \vec{Q}) \frac{\partial}{\partial E} \left[\frac{1}{E} \frac{\partial (f_e + f_{\bar{e}})}{\partial E} \right] . \tag{4.14}$$

When this is substituted in Eq. (4.10), the second term gives zero by symmetric integration, and we obtain

$$I_1(\omega, Q) = -\frac{u \cdot v}{4} \frac{q^2}{Q^2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{E} \frac{\partial^2 (f_e + f_{\bar{e}})}{\partial E^2} \frac{\vec{p} \cdot \vec{Q}}{E\omega - \vec{p} \cdot Q} . \tag{4.15}$$

To evaluate I_1 in the $\vec{Q} \to 0$ limit, we first expand the denominator in the integrand,

$$\frac{1}{E\omega - \vec{p} \cdot \vec{Q}} = \frac{1}{E\omega} \left[1 + \frac{\vec{p} \cdot \vec{Q}}{E\omega} \right] . \tag{4.16}$$

The term proportional to a single power of $\vec{p} \cdot \vec{Q}$ integrates to zero, while in the quadratic term we can put $(\vec{p} \cdot \vec{Q})^2 \to \frac{1}{3} p^2 Q^2$. In this way, in analogy with Eq. (4.7), we obtain the approximate formulas

$$I_1 = \begin{cases} -\frac{u \cdot v}{4} \int \frac{d^3p}{(2\pi)^3} \frac{1}{E} \frac{\partial^2 (f_e + f_{\bar{e}})}{\partial E^2} & (\omega \ll \bar{v}Q) \\ \\ -\frac{u \cdot v}{4} \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3E^3} \frac{\partial^2 (f_e + f_{\bar{e}})}{\partial E^2} & (\omega \gg \bar{v}Q) . \end{cases} \tag{4.17}$$

In what follows we give explicit results for several useful cases.

A.   Non-relativistic, non-degenerate gas

For a classical and non-relativistic gas, using Eq. (3.47) and $f_{\bar{e}} \simeq 0$, Eqs. (4.7) and (4.17) yield

$$I_1 = -u \cdot v \, I_2 , \tag{4.18}$$

with

$$I_2 = n_e \times \begin{cases} \frac{\beta^2}{8m_e} & (\omega \ll \bar{v}Q) \\ \\ \frac{\beta}{8m_e^2} & (\omega \gg \bar{v}Q) , \end{cases} \tag{4.19}$$

where n_e is the total number density of electrons.

B. Relativistic, non-degenerate gas

In this case we also assume for simplicity that $f_{\bar{e}} \simeq f_e$. Then, for I_1, we use Eq. (3.47) twice, and in the limit $m_e \to 0$

$$I_1 = -\frac{u \cdot v}{4\pi^2} \times \begin{cases} 1 & (\omega \ll \bar{v}Q) \\ \\ \frac{1}{3} & (\omega \gg \bar{v}Q). \end{cases} \tag{4.20}$$

For I_2 we have to be careful because the integral is not defined for $m_e = 0$. As we show below, the result is

$$I_2 = \frac{1}{4\pi^2}, \tag{4.21}$$

independently of whether $\omega \ll \bar{v}Q$ or $\omega \gg \bar{v}Q$. To illustrate how we have proceeded let us consider the $\omega \gg \bar{v}Q$ case in some detail. By carrying out the angular integration, followed by an integration by parts, from Eq. (4.7) we obtain the formula

$$I_2 = \frac{m_e^2}{8\pi^2} \int_0^\infty dp\, \frac{(f_e + f_{\bar{e}})}{E^3}. \tag{4.22}$$

By making the change of variable $p = m_e \tan\theta$,

$$\begin{aligned} \int dp\, \frac{f_e + f_{\bar{e}}}{E^3} &= \frac{1}{m_e^2} \int d\theta\,(f_e + f_{\bar{e}}) \cos\theta \\ &= \frac{2}{m_e^2} \int d\theta\, e^{-\beta m \sec\theta} \cos\theta, \end{aligned} \tag{4.23}$$

which we evaluate by using the Taylor expansion of the exponential. In this way we obtain

$$\int_0^\infty dp\frac{f_e + f_{\bar{e}}}{E^3} = \frac{2}{m_e^2}[1 + O(m_e)], \tag{4.24}$$

and substituting it in Eq. (4.22) we arrive at Eq. (4.21). In similar fashion, for $\omega \ll \bar{v}Q$,

$$I_2 = -\frac{m_e^2}{8\pi^2} \int dp\, \frac{1}{E} \frac{\partial}{\partial E} \left(\frac{f_e + f_{\bar{e}}}{E}\right). \tag{4.25}$$

Using Eq. (3.47) and then making the change of variables as above, we obtain

$$\int dp\, \frac{1}{E} \frac{\partial}{\partial E} \left(\frac{f_e + f_{\bar{e}}}{E}\right) = -\frac{2}{m_e^2}[1 + O(m_e)], \tag{4.26}$$

which leads to Eq. (4.21).

C. Degenerate gas

For a degenerate gas, whether it is relativistic or not,

$$\begin{aligned} f_e &= \theta(E_F - E), \\ f_{\bar{e}} &\simeq 0, \end{aligned} \tag{4.27}$$

where $E_F = \sqrt{p_F^2 + m_e^2}$ is the Fermi energy, with

$$p_F = (3\pi^2 n_e)^{1/3} \tag{4.28}$$

being the Fermi momentum. Then we obtain in this case

$$I_1 = -\left(\frac{u \cdot v}{8\pi^2}\right) \times \begin{cases} \frac{E_F}{p_F} & (\omega \ll \bar{v}Q) \\ \\ \frac{p_F}{E_F}\left[1 - \frac{2p_F^2}{3E_F^2}\right] & (\omega \gg \bar{v}Q), \end{cases}$$

$$I_2 = \left(\frac{1}{8\pi^2}\right) \times \begin{cases} \frac{E_F}{p_F} & (\omega \ll \bar{v}Q) \\ \\ \frac{p_F}{E_F} & (\omega \gg \bar{v}Q). \end{cases} \tag{4.29}$$

Furthermore, in the non-relativistic (NR) or the extremely relativistic (ER) case, this yields

$$I_1^{(NR)} = -u \cdot v I_2^{(NR)} = -\left(\frac{u \cdot v}{8\pi^2}\right) \times \begin{cases} \frac{m_e}{p_F} & (\omega \ll \bar{v}Q) \\ \\ \frac{p_F}{m_e} & (\omega \gg \bar{v}Q) \, , \end{cases} \tag{4.30}$$

$$I_1^{(ER)} = -\left(\frac{u \cdot v}{8\pi^2}\right) \begin{cases} 1 & (\omega \ll \bar{v}Q) \\ \\ \frac{1}{3} & (\omega \gg \bar{v}Q) \, , \end{cases} \tag{4.31}$$

and

$$I_2^{(ER)} = \frac{1}{8\pi^2} \tag{4.32}$$

independently of whether $\omega \ll \bar{v}Q$ or $\omega \gg \bar{v}Q$.

<h2 style="text-align:center">D. Explicit formulas for $\pi_P^{(e\nu)}$ and $\pi_P'^{(e\nu)}$</h2>

The formulas that we have obtained for $I_{1,2}$ allow us to evaluate $\pi_P^{(e\nu)}$ and $\pi_P'^{(e\nu)}$ for typical situations of interest in physical applications. We specifically consider the regime

$$\omega \gg \bar{v}Q \, , \tag{4.33}$$

which will be of special interest to us in Section VI B. Thus, for example, if the electron gas is degenerate,

$$\begin{aligned} \pi_P^{(e\nu)} &= -\left(\frac{e^2}{3\pi^2}\right) \frac{\lambda_A p_F^3}{E_F^3} Q \, u \cdot v \, , \\ \pi_P'^{(e\nu)} &= -\left(\frac{e^2}{2\pi^2}\right) \frac{\lambda_A p_F}{E_F} Q \, . \end{aligned} \tag{4.34}$$

These expressions hold whether the electrons are relativistic or not. On the other hand, for a classical and relativistic electron gas,

$$\begin{aligned} \pi_P^{(e\nu)} &= -\left(\frac{2e^2}{3\pi^2}\right) \lambda_A Q \, u \cdot v \, , \\ \pi_P'^{(e\nu)} &= -\left(\frac{e^2}{\pi^2}\right) \lambda_A Q \, . \end{aligned} \tag{4.35}$$

Regarding the value of λ_A, from Eq. (2.2),

$$\lambda_A = \frac{G_F}{\sqrt{2}} \left[(n_{\nu_\tau} - n_{\bar{\nu}_\tau}) + (n_{\nu_\mu} - n_{\bar{\nu}_\mu}) - (n_{\nu_e} - n_{\bar{\nu}_e}) \right] \, . \tag{4.36}$$

Denoting by μ_{ν_x} the chemical potential of the neutrino of flavor $x = e, \mu, \tau$, the approximate formulas

$$\lambda_A = \frac{G_F}{6\sqrt{2}} T^3 \left(\xi_{\nu_\tau} + \xi_{\nu_\mu} - \xi_{\nu_e} \right) \qquad (\mu_{\nu_x} \ll T) \tag{4.37}$$

or

$$\lambda_A = \frac{G_F}{6\sqrt{2}\pi^2} \left(\mu_{\nu_\tau}^3 + \mu_{\nu_\mu}^3 - \mu_{\nu_e}^3 \right) \qquad (\mu_{\nu_x} \gg T) \, , \tag{4.38}$$

can be useful.

E. Discussion

It should be noted that the contributions $\pi_P^{(e\nu)}$ and $\pi_P'^{(e\nu)}$ that we have calculated in this work, have a very different kinematic dependence on the photon momentum q^μ if we compare them with the term $\pi_P^{(\nu)}$ that was calculated in Ref. [17] and the analogous quantity calculated in Ref. [18]. In particular, $\pi_P^{(\nu)}$, which does not depend explicitly on the electron distribution, is proportional to q^2, as we have indicated in Eq. (3.18). For the purpose of determining the photon dispersion relations, since q^2 is of the order of the plasma frequency squared, the value of $\pi_P^{(\nu)}$ turns out to be proportional to the electron density and in fact comparable to the values of $\pi_P^{(e\nu)}$ and $\pi_P'^{(e\nu)}$. In the absence of the electrons, $\pi_P^{(e\nu)}$ and $\pi_P'^{(e\nu)}$ are of course zero and, since $q^2 \simeq 0$, $\pi_P^{(\nu)}$ is negligible. Thus, in a pure neutrino gas the term calculated in Ref. [18], although it is of higher order in $1/M_W^2$, is the dominant one. On the other hand, in applications such as the one that we consider in Section VI B, in which the relevant kinematic limit is $q \to 0$, corresponding to the long wavelength and static regime, $\pi_P^{(\nu)}$ does not contribute and $\pi_P^{(e\nu)}$ and $\pi_P'^{(e\nu)}$ are the only relevant ones.

V. DISPERSION RELATIONS

In the presence of an external current $j_\mu^{(ext)}$, the electromagnetic potential in the medium is determined from the field equation, which in momentum space is

$$\left[-q^2 \tilde{g}_{\mu\nu} + \pi_{\mu\nu}\right] A^\nu = j_\mu^{(ext)}. \tag{5.1}$$

The photon dispersion relations are determined by finding the solutions of the homogeneous equation, i.e., with $j_\mu^{(ext)} = 0$. To specify the various modes it is convenient to introduce the basis vectors $\epsilon_{1,2,3}^\mu$, which we defined as follows.

For a given photon momentum $\vec{Q}$, we define the unit vectors $\hat{e}_i$ $(i = 1, 2, 3)$ by writing

$$\vec{Q} \equiv Q\hat{e}_3, \tag{5.2}$$

with $\hat{e}_{1,2}$ chosen such that

$$\begin{aligned}
\hat{e}_1 \cdot \hat{e}_3 &= e_1 \cdot \hat{e}_3 = 0, \\
\hat{e}_2 &= \hat{e}_3 \times \hat{e}_1.
\end{aligned} \tag{5.3}$$

For the problem that we are considering, without loss of generality, we can choose the vectors $\hat{e}_{1,2}$ such that $\vec{V}$ lies in the $1, 3$ plane. Thus, the unit vector

$$\hat{V} = \frac{\vec{V}}{V}, \tag{5.4}$$

has the decomposition

$$\hat{V} = \cos\theta\,\hat{e}_3 + \sin\theta\,\hat{e}_1. \tag{5.5}$$

where

$$\cos\theta = \hat{Q} \cdot \hat{V}. \tag{5.6}$$

The vectors $\epsilon_{1,2,3}^\mu$ are then defined by

$$\begin{aligned}
\epsilon_1^\mu &= (0, \hat{e}_1), \\
\epsilon_2^\mu &= (0, \hat{e}_2), \\
\epsilon_3^\mu &= \frac{1}{\sqrt{q^2}}(Q, \omega\hat{e}_3),
\end{aligned} \tag{5.7}$$

which form a basis of vectors orthogonal to q^μ. They satisfy the relations

$$\begin{aligned}
R_{\mu\nu}\epsilon_3^\nu = P_{\mu\nu}\epsilon_3^\nu = Q_{\mu\nu}\epsilon_{1,2}^\nu &= 0, \\
R_{\mu\nu}\epsilon_i^\nu &= \epsilon_{i\mu}, \qquad (i = 1, 2), \\
Q_{\mu\nu}\epsilon_3^\nu &= \epsilon_{3\nu}, \\
P_{\mu\nu}\epsilon_1^\nu &= i\epsilon_{2\mu}, \\
P_{\mu\nu}\epsilon_2^\nu &= -i\epsilon_{1\mu},
\end{aligned} \tag{5.8}$$

from which we can immediately read off the matrix elements of the tensors R, Q, P between any pair of the basis vectors $\epsilon^\mu_{1,2,3}$. In addition, the only non-zero matrix elements of the tensor P' are given by

$$\epsilon^\mu_2 \epsilon^\nu_3 P'_{\mu\nu} = -\epsilon^\mu_3 \epsilon^\nu_2 P'_{\mu\nu} = -\frac{i\sqrt{q^2}}{Q} V \sin\theta \,,$$

$$\epsilon^\mu_1 \epsilon^\nu_2 P'_{\mu\nu} = -\epsilon^\mu_2 \epsilon^\nu_1 P'_{\mu\nu} = -\frac{i\omega}{Q} V \cos\theta \,. \tag{5.9}$$

To find the propagating modes, we express the polarization vectors in the form

$$\xi^\mu = \sum_{i=1}^3 \alpha_i \epsilon^\mu_i \,, \tag{5.10}$$

where the coefficients α_i and the corresponding dispersion relations are to be found by solving the equation

$$\left[-q^2 \tilde{g}_{\mu\nu} + \pi_{\mu\nu} \right] \xi^\nu = 0 \,. \tag{5.11}$$

With the help of the relations given in Eqs. (5.8) and (5.9), this equation can be written in matrix notation

$$(q^2 - \Pi)\alpha = 0 \,, \tag{5.12}$$

where

$$\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \,, \tag{5.13}$$

and

$$\Pi = \begin{pmatrix} \pi_T & -i\pi_P + i\frac{\omega}{Q} V \pi'_P \cos\theta & 0 \\ i\pi_P - i\frac{\omega}{Q} V \pi'_P \cos\theta & \pi_T & i\frac{\sqrt{q^2}}{Q} V \pi'_P \sin\theta \\ 0 & -i\frac{\sqrt{q^2}}{Q} V \pi'_P \sin\theta & \pi_L \end{pmatrix} \,. \tag{5.14}$$

In what follows we consider some particular cases of this equation, whose solutions reveal the structure and the main features of more general ones.

A. $\vec{V} = 0$ case

In this case the π'_P term in Eq. (3.44) is absent, and consequently the dispersion relations have the same form as those studied earlier in Ref. [15]. The longitudinal mode has the dispersion relation

$$\omega^2 - Q^2 - \pi_L = 0 \,, \tag{5.15}$$

and corresponding polarization vector

$$\xi^\mu_L = \epsilon^\mu_3 \,. \tag{5.16}$$

The polarization vectors of the transverse modes are then given by

$$\xi^\mu_\pm = \frac{1}{\sqrt{2}} (\epsilon^\mu_1 \pm i\epsilon^\mu_2) \,, \tag{5.17}$$

with the corresponding dispersion relations given by

$$\omega^2 - Q^2 - (\pi_T \pm \pi_P) = 0 \,, \tag{5.18}$$

respectively.

B. $\vec{V} \neq 0$ case

The presence of the π'_P term in Eq. (3.44) in general modifies the picture described above, and can give rise to new effects. This is due to the fact that the existence of the velocity vector $\vec{V}$ of the neutrino gas breaks the three-dimensional isotropy of the medium, for example not too differently from the way in which it would be broken by the presence of an external magnetic field. We consider two special situations.

1. Propagation parallel to $\vec{V}$

In this case $\sin\theta = 0$, and

$$\Pi = \begin{pmatrix} \pi_T & -i\pi_P + i\frac{\omega}{Q}V\pi'_P & 0 \\ i\pi_P - i\frac{\omega}{Q}V\pi'_P & \pi_T & 0 \\ 0 & 0 & \pi_L \end{pmatrix} . \tag{5.19}$$

Therefore, the longitudinal mode is unaffected, while for the transverse modes the polarization vectors are the same as in Eq. (5.17) but the dispersion relations are now given by

$$\omega^2 - Q^2 - \left(\pi_T + \pi_P - \frac{\omega}{Q}V\pi'_P\right) = 0,$$

$$\omega^2 - Q^2 - \left(\pi_T - \pi_P + \frac{\omega}{Q}V\pi'_P\right) = 0, \tag{5.20}$$

for $\xi_\pm$, respectively.

2. Propagation perpendicular to $\vec{V}$

In this case, setting $\theta = \pi/2$,

$$\Pi = \begin{pmatrix} \pi_T & -i\pi_P & 0 \\ i\pi_P & \pi_T & i\frac{\sqrt{q^2}}{Q}V\pi'_P \\ 0 & -i\frac{\sqrt{q^2}}{Q}V\pi'_P & \pi_L \end{pmatrix} . \tag{5.21}$$

Therefore the modes are neither purely longitudinal nor transverse to $\vec{Q}$. Finding the general solution in this case is a cumbersome process. In some circumstances it may be appropriate to seek approximate formulas for the dispersion relations and polarization vectors as a perturbative expansion in $V\pi'_P$, and that can be extended to other values of θ as well.

Some applications of the optical activity effects induced by neutrinos were considered in Ref. [17]. As a rule, the effects tend to be small. Our main intention in this section was to indicate how the photon dispersion relations are modified by the anisotropic effects produced by a non-zero velocity of the neutrino gas relative to the electron background. It can be of interest to carry this further to study the implications of these effects in the context of the specific applications considered in Ref. [17], or similar ones. However, that is outside the scope and focus of the applications that we have considered, to which we now turn our attention.

VI. MACROSCOPIC ELECTRODYNAMICS

Besides modifying the dispersion relations of the propagating modes, the presence of the neutrino gas influence the electromagnetic properties of the system in the static and long wavelength regime. To study them, it is useful to formulate the results or our calculations using the language of macroscopic electrodynamics.

In what follows we will assume that the neutrino gas is at rest with respect to the electron gas, that is

$$v^\mu = u^\mu , \tag{6.1}$$

since this case already brings out the essential consequences of the presence of the neutrino gas. In this case, $P'_{\mu\nu} = 0$, and the photon self-energy takes the form

$$\pi_{\mu\nu} = \pi_T^{(e)} R_{\mu\nu} + \pi_L^{(e)} Q_{\mu\nu} + \pi_P P_{\mu\nu} \,. \tag{6.2}$$

As has been discussed previously[15, 16], this is indeed the most general of the photon self-energy in an isotropic medium, which is the case if $\vec{V} = 0$.

A.   Dielectric function

Introducing the electromagnetic field

$$F_{\mu\nu} = -i(q_\mu A_\nu - A_\mu q_\nu)\,, \tag{6.3}$$

the equation of motion, Eq. (5.1), can be written in the form

$$-iq^\mu F_{\mu\nu} = j_\nu^{(ext)} + j_\nu^{(ind)} \,, \tag{6.4}$$

where

$$j_\mu^{(ind)} = -\pi_{\mu\nu} A^\nu \,. \tag{6.5}$$

In fact, Eq. (6.4) is equivalent to the Maxwell equations, with $j_\mu^{(ind)}$ interpreted as the induced current. For example, take the component of Eq. (6.4) corresponding to the index ν being a spatial index. With the usual definition of the fields,

$$\begin{aligned} \vec{E} &= i\omega\vec{A} - i\vec{Q}A^0\,, \\ \vec{B} &= i\vec{Q} \times \vec{A}\,, \end{aligned} \tag{6.6}$$

which are related by

$$\vec{B} = \frac{1}{\omega}\vec{Q} \times \vec{E}\,, \tag{6.7}$$

the equation is just

$$i\vec{Q} \times \vec{B} + i\omega\vec{E} = \vec{j}^{(ext)} + \vec{j}^{(ind)} \,. \tag{6.8}$$

Moreover, using Eq. (6.6) in Eq. (6.5), the induced current is given in terms of the fields by

$$\vec{j}^{(ind)} = i\omega\left[(1 - \epsilon_l)\vec{E}_l + (1 - \epsilon_t)\vec{E}_t + i\epsilon_p\hat{Q} \times \vec{E}\right]\,, \tag{6.9}$$

where

$$\begin{aligned} 1 - \epsilon_t &= \frac{\pi_T}{\omega^2}\,, \\ 1 - \epsilon_l &= \frac{\pi_L}{q^2}\,, \\ \epsilon_p &= \frac{\pi_P}{\omega^2}\,, \end{aligned} \tag{6.10}$$

and the longitudinal and transverse components of the electric field are defined by

$$\begin{aligned} \vec{E}_l &= \hat{Q}(\hat{Q} \cdot \vec{E})\,, \\ \vec{E}_t &= \vec{E} - \vec{E}_l\,. \end{aligned} \tag{6.11}$$

Eq. (6.9) is the most general form of the induced current, involving terms that are linear in the field, and subject only to the assumption of isotropy. The quantities $\epsilon_{t,l}$ in Eq. (6.10) are transverse and longitudinal components of the

dielectric constant of the medium. Alternatively, instead of $\epsilon_{t,l}$, the dielectric and magnetic permeability functions ϵ, μ are introduced by writing the induced current in the equivalent form

$$\vec{j}^{(ind)} = i \left[\omega(1 - \epsilon)\vec{E} + \left(1 - \frac{1}{\mu}\right) \vec{Q} \times \vec{B} + i\frac{\omega^2}{Q}\epsilon_p\vec{B} \right] , \tag{6.12}$$

where we have used Eq. (6.7). Comparing Eqs. (6.9) and (6.12), the relations

$$\begin{aligned} \epsilon &= \epsilon_l \,, \\ \frac{1}{\mu} &= 1 + \frac{\omega^2}{Q^2}(\epsilon_l - \epsilon_t) \,, \end{aligned} \tag{6.13}$$

then follow.

These equations can of course be used to discuss the dispersion relations of the propagating modes and related effects. However we do not proceed any further in this direction since that would essentially reproduce what we have already considered in Section V. Instead we turn our attention to another kind of effect that can arise due to the presence of the neutrino gas, which can be described on the basis of these equations together with the results we have obtained.

B. Evolution of magnetic fields

Here we consider the evolution of an initial magnetic perturbation in this medium. In the absence of any external sources, the equation for the magnetic field, Eq. (6.8), is

$$i\vec{Q} \times \vec{B} + i\omega\vec{E} = \vec{j}^{(ind)} \,, \tag{6.14}$$

which using Eq. (6.12) we can write in the form

$$\epsilon\omega\vec{E} + \frac{1}{\mu}\vec{Q} \times \vec{B} + i\gamma\vec{B} = 0 \,, \tag{6.15}$$

with

$$\gamma = -\frac{\omega^2}{Q}\epsilon_p = -\frac{\pi_P}{Q} \,. \tag{6.16}$$

Since we are interested in following the evolution of $\vec{B}$, we eliminate $\vec{E}$ from this equation by taking the cross product with $\vec{Q}$ and then using Eq. (6.7), which yields

$$\epsilon\omega^2\vec{B} - \frac{1}{\mu}Q^2\vec{B} + i\gamma\vec{Q} \times \vec{B} = 0 \,. \tag{6.17}$$

We obtain the corresponding equation in coordinate space by taking the long wavelength limit, $\omega \gg \bar{v}Q$, and making the quasi-static approximation, $\omega \to 0$. In this limit

$$\epsilon \to 1 + \frac{i\sigma}{\omega} \,, \tag{6.18}$$

where σ is the conductivity of the medium. Therefore the equation becomes

$$i\omega\sigma\vec{B} - \frac{1}{\mu}Q^2\vec{B} + i\gamma\vec{Q} \times \vec{B} = 0 \,, \tag{6.19}$$

or, in coordinate space,

$$\sigma\frac{\partial\vec{B}}{\partial t} = \frac{1}{\mu}\nabla^2\vec{B} + \gamma\vec{\nabla} \times \vec{B} \,. \tag{6.20}$$

Although we are using the same symbols μ and γ, in Eq. (6.20) they stand for the corresponding quantities evaluated in the long wavelength and static limit, as indicated above. In this limit, $\pi_P^{(\nu)}$ does not contribute to γ in Eq. (6.16),

as can be seen from Eq. (3.18). Then using the results for $\pi_P^{(e\nu)}$ summarized in SectionIV D, we can readily determine γ for some particular cases. For example, for the relativistic and non degenerate electron gas,

$$\gamma = \frac{2e^2 \lambda_A}{3\pi^2} , \tag{6.21}$$

while for a degenerate electron gas,

$$\gamma = \frac{e^2 \lambda_A}{3\pi^2} \left(\frac{p_F}{E_F} \right)^3 , \tag{6.22}$$

with λ_A given in Eq. (4.36). For more general cases, γ can be computed from the formula

$$\gamma = -e^2 \lambda_A \left[\int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3E^3} \frac{\partial^2 (f_e + f_{\bar{e}})}{\partial E^2} + m_e^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{E} \frac{\partial}{\partial E} \left(\frac{f_e + f_{\bar{e}}}{E^3} \right) \right] , \tag{6.23}$$

which follows from the expression for $\pi_P^{(e\nu)}$ given in Eq. (3.46), together with the formulas for $I_{1,2}$ given in Eqs. (4.7) and (4.17) for $\omega \gg \bar{v}Q$.

An equation of the same form as Eq. (6.20) was obtained by Semikoz and Sokoloff[20] by very different means, in their work suggesting a new mechanism for the generation of large-scale magnetic fields in the Early Universe as a consequence of the neutrino-plasma interactions. As emphasized in that reference, the equation describes the self-excitation of an almost constant magnetic perturbation.

Our approach is useful in two complementary ways. On one hand, it sheds light on the physical origin of this mechanism. Within our formulation, it is a consequence of the optical activity induced by the interaction of neutrinos with the other background particles. On the other hand, it puts this mechanism on a firm footing from a computational point of view. The optical activity induced by the neutrinos can be characterized by the presence of the π_P (and π_P') terms in the photon self-energy, which also has a definite interpretation in terms of the components of the dielectric function that enter in the macroscopic (Maxwell) equations in the medium. Therefore, by focusing on these quantities, we are able to give well defined formulas for the parameters that are relevant to this effect, in a way that are applicable to a variety of astrophysical and cosmological situations in which the presence of neutrinos is influential. Our work also paves the way to incorporate some corrections that can be important in specific applications, such as the anisotropic effects that can arise if the neutrino gas has a non-zero velocity relative to the electron background. Further studies along these lines are the subject of current work.

VII. CONCLUSIONS

We have studied the electromagnetic properties of an electron background, that contains a neutrino gas which is either at rest or moving as a whole relative to the background. Apart from the well known longitudinal and transverse polarization functions of the photon in a medium, the presence of the neutrinos gives rise to two additional polarization functions, that we denote by π_P and π_P'. We have computed that particular contribution to these two functions, $\pi_P^{(e\nu)}$ and $\pi_P'^{(e\nu)}$, that depends on the neutrino-antineutrino asymmetry in the medium as well as the momentum integral of the electron (and positron) distribution function. The integrals were evaluated for various specific conditions of the electron gas, and explicit formulas that are useful in many situations were given. One of the consequences of a non-zero value of π_P and π_P' is to give rise to birefringence and anisotropic effects in the propagation of a photon through that medium. We analyzed various particular situations to indicate how the anisotropies due to the non-zero velocity of the neutrino gas can affect the optical activity of the system. A non-zero value of π_P and π_P' also has consequences related to the electromagnetic properties of the system at a macroscopic level, and we considered specifically the evolution of a macroscopic magnetic field in this system. We arrived at an equation for the dynamics of the magnetic field, that had been suggested in Ref. [20] as a mechanism for the generation of large-scale magnetic fields in the Early Universe as a consequence of the neutrino-plasma interactions. In this way we established contact between our work and this particular kind of application, which has been of recent interest. The approach we followed, which has been based on the application of Finite Temperature Field Theory, as well as the calculations and results that are presented here, helps to put this subject on a firm footing and to set a basis for carrying out further studies and applications along these lines using powerful calculational techniques.

Acknowledgments

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